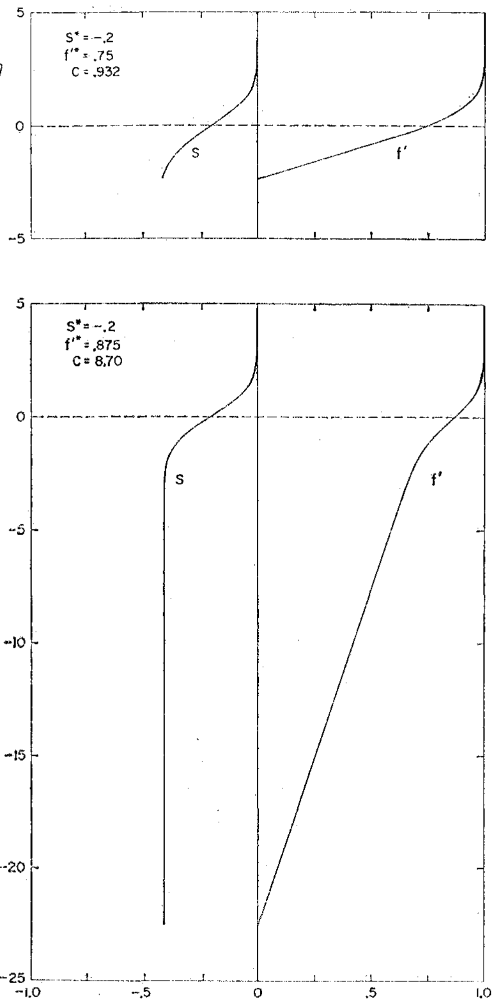

Fig. 2 Boundary-layer profiles.

irrespective of the value of the blowing parameter. Unlike the usual numerical methods, which cease converging to the solution when the blowing parameter increases, the proposed method actually converges to the solution more rapidly as the value of the blowing parameter increases.

The satisfaction of the two-point boundary conditions is a problem common to all means of solving the boundary-layer equations; the method proposed herein should aid in improving the schemes for solving complex boundary-layer problems.

References

[1] Tauber, M., "Atmospheric Entry Into Jupiter," *Journal of Spacecraft and Rockets*, Vol. 6, No. 10, Oct. 1969, pp. 1103–1109.

[2] Libby, P. A., "The Homogeneous Boundary Layer at an Axisymmetric Stagnation Point with Large Rates of Injection," *Journal of the Aerospace Sciences*, Vol. 29, No. 1, Jan. 1962, pp. 48–60.

[3] Aroesty, J. and Cole, J. D., "Boundary Layer Flows with Large Injection Rates," Memo-RM-4620-ARPA, 1965, Rand Corporation, Santa Monica, Calif.

[4] Kubota, T. and Fernandez, F. L., "Boundary Layer Flows with Large Injection and Heat Transfer," *AIAA Journal*, Vol. 6, No. 1, Jan. 1968, pp. 22–28.

[5] Libby, P. A., "On the Numerical Analysis of Stagnation Point Flows with Massive Blowing," *AIAA Journal*, Vol. 8, No. 11, Nov. 1970, pp. 2095–2096.

[6] Pretsch, J., "Analytic Solutions of the Laminar Boundary Layer with Asymptotic Suction and Injection," *Zeitschrift für angewandt Mathematik und Mechanic*, Vol. 24, No. 5, June 1944, pp. 264–267.

[7] Nachtsheim, P. R. and Swigert, P., "Satisfaction of Asymptotic Boundary Conditions in the Numerical Solution of Boundary Layer Equations," *Proceedings of the Ninth Midwestern Mechanics Conference*, The University of Wisconsin, Madison, Aug. 1965, pp. 361–371.

[8] Cohen, C. B. and Reshotko, E., "Similar Solutions for the Compressible Laminar Boundary Layer with Heat Transfer and Pressure Gradient," Rept. 1293, 1956, NACA.

[9] Christian, J. W., Hankey, W. L., and Petty, J. S., "Similar Solutions of the Attached and Separated Compressible Laminar Boundary Layer with Heat Transfer and Pressure Gradient," Rept. 70-0023, Feb. 1970, Aerospace Research Labs., Air Force Systems Command, Wright-Patterson Air Force Base, Ohio.

Hypersonic Strong Interaction Flow over an Inclined Surface

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THE purpose of this Note is to present a solution for the hypersonic strong interaction flow over an inclined surface. An asympotic expansion in powers of the hypersonic interaction parameter $\bar{\chi}$ is used to reduce the boundary-layer equations to a sequence of ordinary differential equations. The scheme was originally suggested in Refs. 1 and 2. References 3 and 4 give the zero-order solutions for $Pr = 1$. In Ref. 5 a two-term solution, which corresponds to the zeroth and the second-order solutions of the present scheme, was obtained by using the Karman-Pohlhausen integral method for the flow over an insulated flat plate. Here some results are presented for $Pr = 1$ and 0.72 with different thermal conditions on the plate.

Let the physical variables be denoted by the superscript* and let the subscripts ∞ and e represent the conditions in the freestream and in the inviscid outer flow (Fig. 1). Theconditions in the boundary layer are represented without any subscript. The dependent variables are nondimensionalized with respect to their freestream values, and the independent variables x^*,y^* with respect to a characteristic length dimension L. The nondimensionalized quantities are represented without any superscript. The gas is assumed to have constant c_p,γ and Pr and obey a linear viscosity-temperature relation $\mu = CT$, C being a constant.

The pressure distribution on the surface and the boundary-layer displacement thickness are given by[1,2]:

$$p(x) = \frac{p^*(x^*)}{p_\infty^*} = p_0\bar{\chi}\left[1 + \frac{p_1K_b}{\bar{\chi}^{1/2}} + \frac{p_2 + p_3K_b^2}{\bar{\chi}} + 0(\bar{\chi}^{-3/2})\right] \quad (1)$$

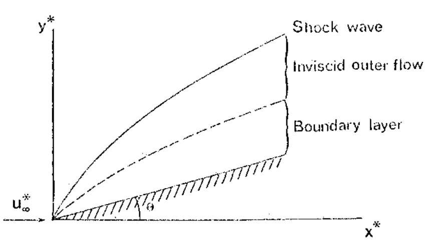

Fig. 1 Flowfield above an inclined plate in hypersonic viscous flow.

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Table 1 Constants $p_i, \delta_i (i = 0,1,2,3)$ of Eqs. (1) and (2) ($\gamma = 1.4$)

Pr	Thermal condition on the surface	p_0	p_1	p_2	p_3	δ_0	δ_1	δ_2	δ_3
1.0	$H_w = 0.0$	0.14110 (0.1485)[a]	2.8385	5.3434	4.6565	0.38641 (0.3967)[a]	−1.5235	−3.0037	0.79265
1.0	$H_w = 0.5$	0.32918	1.8481	2.2944	1.9947	0.59020	−1.0013	−1.2863	0.34224
1.0	$H_w = 1.0$	0.50892 (0.51)[b]	1.4969	1.5097 (1.48)[b]	1.1443	0.73385 (0.738)[a]	−0.80134	−0.82430	0.17524
0.72	$H_w = 0.0$	0.12368	3.0009	6.0238	4.7967	0.36177	−1.6388	−3.4484	0.76358
0.72	$H_w = 0.5$	0.33702	1.8134	2.2196	1.8510	0.59719	−0.99449	−1.2628	0.30868
0.72	$H_w = 1.0$	0.54042	1.4451	1.4114	1.1813	0.75622	−0.78046	−0.77936	0.19777
0.72	$H'(0) = 0.0$	0.47311	1.5494	1.6171	1.3171	0.70756	−0.83227	−0.88877	0.23149

[a] From Ref. 4.
[b] From Ref. 5; for $Pr = 1.0$, the conditions $H_w = 1.0$ and $H'(0) = 0.0$ are identical.

$$\frac{\delta^*(x^*)}{x^*} = \delta_0 \frac{\bar{\chi}^{1/2}}{M_\infty}\left[1 + \frac{\delta_1 K_b}{\bar{\chi}^{1/2}} + \frac{\delta_2 + \delta_3 K_b^2}{\bar{\chi}} + 0(\bar{\chi}^{-3/2})\right] \quad (2)$$

where $M_\infty(\gg 1)$ is the freestream Mach number, $K_b = M_\infty\theta$, and $\bar{\chi} = M_\infty^3(C/Re_{\infty,x^*})^{1/2} = M_\infty^3(C\mu_\infty^*/\rho_\infty^* u_\infty^* x^*)^{1/2}$ is the hypersonic interaction parameter. The constants p_0,p_1,p_2,p_3 and $\delta_0,\delta_1,\delta_2,\delta_3$, which are not known a priori, are not independent of each other. Using the tangent wedge approximation the following relations are obtained for these constants:

$$p_0 = \tfrac{9}{32}\gamma(\gamma + 1)\delta_0^2 \quad (3a)$$

$$p_1 = \tfrac{8}{3}(\delta_1 + 1/\delta_0) \quad (3b)$$

$$p_2 = \tfrac{10}{3}\delta_2 + \tfrac{32}{9}(3\gamma + 1)/[\gamma(\gamma + 1)^2\delta_0^2] \quad (3c)$$

$$p_3 = \tfrac{10}{3}\delta_3 + \tfrac{16}{9}(\delta_1 + 1/\delta_0)^2 \quad (3d)$$

The boundary-layer equations transform to

$$\bar{p}f_{\eta\eta\eta} + ff_{\eta\eta} + 4x(f_x f_{\eta\eta} - f_{\eta x}f_\eta) = 2\beta\frac{x}{p}\frac{dp}{dx}(H - f_\eta^2) \quad (4)$$

$$\frac{\bar{p}}{Pr}H_{\eta\eta} + H_\eta f + 4x(f_x H_\eta - f_\eta H_x) = -2\bar{p}\left(1 - \frac{1}{Pr}\right)[(f_{\eta\eta})^2 + f_\eta f_{\eta\eta\eta}] \quad (5)$$

where

$$\bar{p} = p/p_0\bar{\chi}, \quad \beta = (\gamma - 1)/\gamma,$$

$$\eta = \left[\int_0^y \rho dy/2(p_0\bar{\chi}x)^{\frac{1}{2}}\right](\rho_\infty^* u_\infty^* L/\mu_\infty^* C^{\frac{1}{2}})$$

and

$$f(x,\eta) = [\psi/2(p_0\bar{\chi}x^{\frac{1}{2}})(\rho_\infty^* u_\infty^* L/\mu_\infty^* C^{\frac{1}{2}})$$

ψ being the stream function defined by $\rho u = \partial\psi/\partial y$, $\rho v = -\partial\psi/\partial x$.

In view of the expansions for p and δ^*/x^*, the following expansions for f and H seem to be appropriate:

$$f(x,\eta) = f_0(\eta) + \frac{f_1(\eta)K_b}{\bar{\chi}^{1/2}} + \frac{f_2(\eta) + f_3(\eta)K_b^2}{\bar{\chi}} + 0(\bar{\chi}^{-3/2}) \quad (6)$$

$$H(x,\eta) = H_0(\eta) + \frac{H_1(\eta)K_b}{\bar{\chi}^{1/2}} + \frac{H_2(\eta) + H_3(\eta)K_b^2}{\bar{\chi}} + 0(\bar{\chi}^{-3/2}) \quad (7)$$

If expansions (1), (2), (6), and (7) are now inserted in Eqs. (4) and (5) and terms of the same order on two sides are equated, then a sequence of coupled ordinary differential equations for $f_i,H_i (i = 0,1,2,3)$ is obtained. The equations for f_0,H_0 are

$$f_0''' + f_0 f_0'' + \beta(H_0 - f_0'^2) = 0 \quad (8a)$$

$$H_0'' + Prf_0 H_0' - 2(1 - Pr)[f_0''^2 + f_0' f_0'''] = 0 \quad (8b)$$

The differential equations for $f_i,h_i (i = 1,2,3)$ are linear with the unknown constants p_1,p_2,p_3 in the homogeneous part. These constants are eliminated if a further change of variables is made:

$$f_1 = p_1\bar{f}_1 \quad H_1 = p_1\bar{H}_1 \quad (9a)$$

$$f_2 = p_2\bar{f}_2 \quad H_2 = p_2\bar{H}_2 \quad (9b)$$

$$f_3 = p_3\bar{f}_2 + p_1^2\bar{f}_3 \quad H_3 = p_3\bar{H}_2 + p_1^2\bar{H}_3 \quad (9c)$$

The new variables $\bar{f}_i,\bar{H}_i (i = 1,2,3)$ now satisfy

$$L_i^{(1)}(\bar{f}_i) + \beta(\bar{H}_i) = F_i^{(1)}(\eta) \quad (10a)$$

$$L_i^{(2)}(\bar{H}_i) + L_i^{(3)}(\bar{f}_i) = F_i^{(2)}(\eta) \quad (10b)$$

where the linear operators $L_i^{(1)},L_i^{(2)},L_i^{(3)}$ are defined by

$$L_i^{(1)} = \frac{d^3}{d\eta^3} + f_0\frac{d^2}{d\eta^2} - (a_i + 2\beta)f_0'\frac{d}{d\eta} + b_i f_0'' \quad (11)$$

$$L_i^{(2)} = \frac{d^2}{d\eta^2} + Prf_0\frac{d}{d\eta} - a_i Prf_0' \quad (12)$$

$$L_i^{(3)} = -2(1 - Pr)\left(f_0'\frac{d^3}{d\eta^3} + 2f_0''\frac{d^2}{d\eta^2} + f_0'''\frac{d}{d\eta}\right) + b_i PrH_0' \quad (13)$$

$a_i = 1, b_i = 2$ for $i = 1$; $a_i = 2$; $b_i = 3$ for $i = 2,3$; and the functions $F_i^{(1)}$ and $F_i^{(2)}$ are

$$F_i^{(1)}(\eta) = c_i\beta(H_0 - f_0'^2) + f_0''f_0 \quad (14)$$

$c_i = 1.5$ for $i = 1$; $c_i = 2$ for $i = 2$;

$$F_i^{(1)}(\eta) = (\bar{f}_1'' - f_0'')(f_0 - 2\bar{f}_1) - \bar{f}_1'[f_0' - (1 + \beta)\bar{f}_1'] + \tfrac{3}{2}\beta(\bar{H}_1 - 2f_0'\bar{f}_1') - 2\beta(H_0 - f_0'^2) \quad \text{for } i = 3 \quad (15)$$

$$F_i^{(2)} = PrH_0'f_0 \quad \text{for } i = 1,2 \quad (16)$$

$$F_i^{(2)} = Pr(\bar{H}_1' - H_0')(f_0 - 2\bar{f}_1) + Pr\bar{H}_1(\bar{f}_1' - f_0') + 2(1 - Pr)[(\bar{f}_1'')^2 + \bar{f}_1\bar{f}_1'''] \quad \text{for } i = 3 \quad (17)$$

The boundary conditions are given by at

$\eta = 0$: $f_i = 0 = f_i'$, $(i = 0,1,2,3)$;

$H_0 = H_w$(= constant) and $H_i = 0 (i = 1,2,3)$ for heat transfer at the surface,

or $H_i' = 0 (i = 0,1,2,3)$ for insulated plate

At $\eta \to \infty$

$$f_0' = 1 \quad (18a)$$

$$f_i' = 0 \quad (i = 1,2,3) \quad (18b)$$

$$H_0 = 1 \tag{18c}$$

$$H_i = 0 \quad (i = 1,2,3) \tag{18d}$$

These are boundary value problems with three conditions given at $\eta = 0$ and the other two at $\eta \to \infty$. The zero-order equations (8) are solved by an iterative method. The higher order variables are solved in three parts and a linear combination of these three solutions is made to satisfy the $\eta \to \infty$ conditions. Hamming's Predictor-corrector method is used with a stepsize of 2^{-7} and $\eta \to \infty$ conditions are satisfied at $\eta = 5$ within a tolerence of 10^{-3}.

The boundary-layer displacement thickness $\delta^*(x^*)$, defined by

$$\delta^*(x^*) = \int_0^{y_e^*} \left(1 - \frac{\rho^* u^*}{\rho_e^* u_e^*}\right) dy^*$$

after some algebraic manupulation gives

$$\frac{\delta^*}{x^*} = \frac{\gamma - 1}{(p_0)^{1/2}} \frac{\bar{\chi}^{1/2}}{M_\infty} \left(I_0 + p_1(I_1 - I_0) \frac{K_b}{\bar{\chi}^{1/2}} + p_2(I_2 - I_0) + [p_3(I_2 - I_0) + p_1^2(I_3 - I_1 + I_0)]K_b^2/\bar{\chi} \right) \tag{19}$$

where

$$I_0 = \int_0^\infty (H_0 - f_0'^2) d\eta \tag{20a}$$

$$I_1 = \int_0^\infty (\bar{H}_1 - 2f_0'\bar{f}_1') d\eta \tag{20b}$$

$$I_2 = \int_0^\infty (\bar{H}_2 - 2f_0'\bar{f}_2') d\eta \tag{20c}$$

$$I_3 = \int_0^\infty (\bar{H}_3 - 2f_0'\bar{f}_3' - \bar{f}_1'^2) d\eta \tag{20d}$$

Comparing Eq. (19) with Eq. (2) and using Eq. (3), one obtains

$$p_0 = \tfrac{3}{4}(\gamma - 1)[\gamma(\gamma + 1)/2]^{1/2} I_0 \tag{21a}$$

$$p_1 = p_0^{1/2}/(\gamma - 1)(\tfrac{11}{8} I_0 - I_1) \tag{21b}$$

$$p_2 = 32(3\gamma + 1)/3\gamma(\gamma + 1)^2(\gamma - 1)^2 I_0(13I_0 - 10I_2) \tag{21c}$$

$$p_3 = p_1^2(40I_3 - 40I_1 + 43I_0)/(54I_0 - 40I_2) \tag{21d}$$

The integrals I_0, I_1, I_2, I_3 are calculated from the solutions $f_i, H_i (i = 0,1,2,3)$ using Simpson's quadrature formula. The constants $p_i, \delta_i (i = 0,1,2,3)$ are then obtained from Eqs. (21) and (3). Values of $p_i, \delta_i (i = 0,1,2,3)$ are given in Table 1 for the insulated plate case and also for wall heat transfer with $H_w = 0.0$, 0.5 and 1.0 for $Pr = 1.0$ and 0.72. Results available from Refs. 3–5 are also included for comparison.

References

[1] Lees, L. and Probstein, R. F., "Hypersonic Viscous Flow Over a Flat Plate," Rept. 195, 1952, Dept. of Aerospace Engineering, Princeton Univ., Princeton, N.J.

[2] Hayes, W. D. and Probstein, R. F., *Hypersonic Flow Theory*, Academic Press, New York, 1959.

[3] Li, T. Y. and Nagamatsu, H. T., "Shock-Wave Effects on the Laminar Skin Friction of an Insulated Flat Plate at Hypersonic Speeds," *Journal of the Aeronautical Sciences*, Vol. 20, 1953, pp. 345–355.

[4] Li, T. Y. and Nagamatsu, H. T., "Hypersonic Viscous Flow on Noninsulated Flat Plate," *Proceedings from the 4th Midwestern Conference on Fluid Mechanics*, Purdue Univ., 1955, pp. 273–287.

[5] Nagakura, T. and Naruse, H., "An Approximate Solution of the Hypersonic Laminar Boundary-Layer Equations and Its Application," *Journal of the Physical Society of Japan*, Vol. 12, 1957, pp. 1298–1304.

Asymptotic Behavior of the Kalman Filter with Exponential Aging

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Introduction

Often, in applying the theory of recursive least-squares state estimation or Kalman filtering, some artificial means of "aging" data must be used in order that measurements taken long in the past not effect the current state estimates. This is frequently caused by imperfect knowledge of the system under consideration. For example, if one of the parameters to be estimated is assumed constant when, in fact, it may change slowly in some unspecified manner, then estimates of the parameter should be based only on recent data. Several methods of effectively aging data fall easily within the framework of the Kalman filter.[1] One method is to assume a certain amount of input (or state disturbance) noise. This will allow the state estimates to deviate from the assumed dynamics. This method has been discussed by Peschon and Larson[2] and by Manchee.[3] Another method is to introduce a weighting function into the measurement covariance matrix such that the assumed variance of the measurements increases with the age of the data. The choice of an exponential aging factor has computational advantages. Such a technique has been discussed as early as 1964 by Fagin,[4] and very recently by Tarn and Zaborsky.[5]

Because of the importance of the Kalman filter with exponential aging, it is of interest to study the effect of the aging factor in some detail. Since the aging factor is merely an artifice to improve the filter performance, the problem of choosing the aging "time constant," which is a free parameter, cannot be formulated mathematically. The best that can be done is to obtain some analytical results concerning the effect of aging that can be used as an aid to the designer. Some of the required results are presented herein for stationary systems.

Continuous Stationary Systems

Consider the case where there are no random state disturbances. Let

$$\dot{x} = Fx + u, \quad y = Hx + n \tag{1}$$

where x is the unknown state vector, u is the known input vector, y is the known measurement vector, n is the unknown noise vector, and F and H are constant matrices. Suppose that

$$E(n) = 0$$

$$E(n(t_1)n'(t_2)) = R\delta(t_1 - t_2)$$

where $(\ )'$ denotes transpose, $E(\ )$ denotes statistical expectation, $\delta(t)$ is the unit impulse, and R is the (symmetric, positive-definite) noise covariance matrix.

The Kalman filter equations for obtaining the maximum likelihood estimate $\hat{x}$ of $x(t)$, based on $y(\tau)$; $0 \leq \tau \leq t$ are

$$(d/dt)(\hat{x}) = F\hat{x} + u + PH'R^{-1}(y - H\hat{x}) \tag{2a}$$

$$\dot{P} = FP + PF' - PH'R^{-1}HP \tag{2b}$$

where P is the estimate covariance matrix that is symmetric.

Now exponential aging is introduced by changing the noise covariance matrix to R^*:

$$R^*(t) = e^{+at}R(t), \quad a > 0 \tag{3}$$

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